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Non-Critical Light-Cone String

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Abstract

The free non-critical string quantum model is constructed directly in the light-cone variables in the range of dimensions $1 < D < 25$. The longitudinal degrees of freedom are described by an abstract Verma module. The central charge of this module is restricted by the requirement of the closure of the nonlinear realization of the Poincare algebra. The spin content of the model is analysed. In particular for $D = 4$ the explicit formulae for the character generating functions of the open and closed massive strings are given and the spin spectrum of first 12 excited levels is calculated. It is shown that for the space-time dimension in the range $1 < D < 25$ the non-critical light-cone string is equivalent to the critical massive string and to the non-critical Nambu-Goto string.

Introduction

The light-cone description of string dynamics was first analysed long time ago in the fundamental paper of Goddard, Goldstone, Rebbi and Thorn [1], where the covariant quantum Nambu-Goto string model was compared to the quantum model obtained by canonical quantization of light-cone gauge fixed classical theory. It was shown that the light-cone quantum theory was relativistically invariant only in the critical space-time dimension $D = 26$ and for the unit intercept of the leading Regge trajectory $\alpha(0) = 1$. On the other hand the covariant scheme of quantization yields a ghost-free quantum theory in the space-time dimensions $D \leq 26$ [2, 3]. Generically the gauge symmetry breaks down upon quantization and in the subcritical dimensions the quantum theory inevitably contains longitudinal degrees of freedom. In consequence one has more degrees of freedom than in the classical model. Only for $D = 26$ and $\alpha(0) = 1$ the classical symmetry is restored in the sense that the states with longitudinal excitations become null and drop out from physical amplitudes. As far as the free theory is concerned the equivalence of both method of quantization can be shown for the critical dimension and intercept using the results of [4].

The light-cone formalism plays a prominent role in the critical string theory. This is the only known formulation of quantum string theory in terms of physical degrees of freedom. This is also the only framework in which one can explicitly realize the geometric idea of joining-splitting interaction [5]. Finally the S-matrix defined peturbatively within the light-cone approach is automatically unitary. The price one has to pay for this is a non-trivial realization of the Poincare symmetry and a difficult and technically demanding proof of the Poincare invariance of the light-cone S-matrix [5, 6]. It should be stressed however that known proofs of unitarity of the covariant theory are based either on the equivalence of the covariant Polyakov string theory in the critical dimension $D = 26$ with the light-cone Mandelstam theory [7], or on the so called α-invariance in the "covariantized light-cone" BRST formulation [8], which also implies Poincare invariance of the standard light-cone formulation [9].

The aim of the present paper is to show that the light-cone formulation can be extended to non-critical string models. As it was mentioned above such a formulation is essential for constructing and analysing joining-splitting interaction and a unitary perturbative expansion of the S-matrix. All hitherto attempts to construct interaction of non-critical strings were made within the covariant formulation of dual models [10, 11, 12, 13]. In the case of non-critical Polyakov string a straightforard extension of the dual model conformal field theory construction failed due to the so called $c = 1$ barrier [14]. The equivalence of the covariant Polyakov formulation to the Mandelstam light-cone theory is known only for the critical dimension and intercept [7]. This is therefore an open question how the $c = 1$ barrier manifests itself in an explicitly unitary formulation and whether the joining-spliting interaction in such a formulation may lead to a Lorentz covariant theory in subcritical dimensions.

In the present paper we shall restrict ourselves to a detailed analysis of the free theory leaving the main problem of interactions for future publications. The content of the paper is as follows. In Section 1 we construct a quantum free non-critical string model directly in the light cone variables. In this construction the longitudinal degrees of freedom are described by an abstract Verma module. The central charge of this module is restricted by the requirement of the closure of the non-linear realization of the Poincare algebra. Let us note that a similar construction was first considered by Marnelius in the context of non-critical Polyakov string [15]. A peculiar feature of this quantum model is that the corresponding classical system is not Lorentz covariant because of anomalous terms in Poisson bracket algebra of classical Lorentz

generators. These terms are cancelled in the quantum theory by terms arising from normal ordering and one gets a unitary representation of the Poincare algebra. The cancellation takes place only for specific critical values of parameters of this construction.

In Section 2 we analyse the spectrum of the non-critical light-cone string introduced in Section 1. The formula for the character generating function is obtained and its explicit expansion in terms of irreducible characters is derived in the physical space-time dimensions $D = 4$. These results are illustrated by numerical calculations of the spin content of the first 12 excited levels. The corresponding results for the closed massive strings are briefly presented in Section 3.

The rest of the paper is devoted to the relations between various non-critical free string models. We prove equivalence results which imply that the light-cone model of non-critical string constructed in Section 1 is universal to some extend.

In Section 4 we briefly recall the massive string model. As a quantum model it was first constructed long time ago by Chodos and Thorn [10]. It was later considered by Marnelius in the context of non-critical Polakov string [15]. The relation between these two models was also analysed in [16]. A detailed discussion of the classical and quantum theory of the free massive string was recently given in [17].

The classical world sheet action for the massive string is the extension of the BDHP action by the fre Liouville action for an additional dimensionless scalar field:

$$
\begin{aligned}
S[M,g,\varphi,x] \;=\; & -\frac{\alpha}{2\pi}\int_M \sqrt{-g}d^2z\; g^{ab}\partial_a x^\mu \partial_b x_\mu \\
& -\frac{\beta}{2\pi}\int_M \sqrt{-g}d^2z\;\left(g^{ab}\partial_a\varphi\partial_b\varphi + 2R_g\varphi\right) \quad .
\end{aligned}
$$

One of the important differences with respect to the classical Nambu-Goto or BDHP string models is that the Virasoro constraints are of second class in the conformal gauge. This means that one cannot use the classical light-cone gauge in order to parameterize the reduced phase space of the model. In fact solving classical constraints is equivalent to solving a special class of non-linear Hill's equations which makes canonical quantization in the physical variables prohibitively difficult.

Using an alternative covariant method of quantization in which constraints are imposed as conditions for physical states one obtains a family of quantum models characterized by the dimensionless coupling β and the physical intercept $\alpha(0)$. Among all the values of these parameters allowed by the no-ghost theorem the so called critical ones $\beta_{\rm crit} = \frac{25-D}{48}$, $\alpha(0)_{\rm crit} = \frac{D-1}{24}$ are especially interesting. In this case the space $\mathcal{H}$ of physical states contains the largest subspace $\mathcal{H}_0$ of null states. In order to analyse a physical content of the model a tractable parameterization of the quotient

$$
\pi : \mathcal{H} \longrightarrow \mathcal{H}_{\rm phys} \;\; = \mathcal{H}/\mathcal{H}_0 \tag{1}
$$

is required. One possible approach is to find a subspace $\mathcal{H}_{\rm gauge} \subset \mathcal{H}$ such that the projection π restricted to $\mathcal{H}_{\rm gauge}$ is a 1-1 map. Such a subspace provides a section of the fibration (1) and will be called a (quantum) gauge. Indeed the shift by a null state can be regarded as a quantum gauge transformation and the subspace $\mathcal{H}_{\rm gauge}$ can be seen as a gauge slice for this symmetry.

The basic idea in our proof of equivalence of different string models is to regard them as different descriptions of the critical massive string related to different quantum gauges. In particular in Section 5 we use the quantum light-cone gauge to prove the equivalence of the critical massive string with the non-critical light-cone string of Section 1. Our method is

a simplification and a generalisation of the method used in a similar context in the critical string theory [4]. In Section 6 we consider another gauge slice which is stable with respect to the Poincare group action. It provides a simple proof of the old conjecture [15] that the critical massive string is equivalent to the non-critical Nambu-Goto and Polyakov strings.

Finally for the sake of completeness some detailes of the DDF construction are presented in Appendix.

1 Non-Critical Light-Cone String

In this section we construct a quantum string model in the range of space-time dimensions $1 < D < 25$ directly in the light-cone variables.

The starting point of the construction is a choice of a light-cone frame in the D-dimensional flat Minkowski target space. It consists of a pair k, k' of light-like vectors $k^2 = 0 = k'^2$ satisfying $k{\cdot}k' = -1$, and a complementary set of transverse vectors $\{e_i\}_{i=1}^{D-2}$ forming an orthonormal basis of the subspace orthogonal to both k and k'. Let us denote by

$$\begin{aligned} P^+ &= k\cdot P \quad , \quad x^- = k'\cdot x \quad , \\ P^i &= e^i\cdot P \quad , \quad x^i = e^i\cdot x \quad , \quad i = 1,..,D-2 \quad , \end{aligned}$$

the self-adjoint operators corresponding to the light-cone components of momentum and position and satisfying the standard commutation relations

$$[P^i, x^j] \;=\; i\delta^{ij} \quad , \quad [P^+, x^-] \;=\; -i \quad .$$

For each $p^+, \overline{p} = \sum p^i e^i$ we define the Fock space $\mathcal{F}^{\rm T}(p^+, \overline{p})$ generated by the infinite algebra of transverse excitation operators:

$$[\alpha^i_m, \alpha^j_n] = m\delta^{ij}\delta_{m,-n} \quad , \quad (\alpha^i_m)^\dagger \;=\; \alpha^i_{-m} \quad , \quad m,n \in \mathbb{Z} \quad ,$$

out of the unique vacuum state $\Omega(p^+, \overline{p})$ satisfying

$$\begin{aligned} \alpha^i_0\Omega(p^+, \overline{p}) &= \tfrac{1}{\sqrt{\alpha}}p^i\Omega(p^+, \overline{p}) \quad , \\ \alpha^+_0\Omega(p^+, \overline{p}) &= \tfrac{1}{\sqrt{\alpha}}p^+\Omega(p^+, \overline{p}) \quad , \\ \alpha^i_m\Omega(p^+, \overline{p}) &= 0 \quad , \quad m > 0 \quad . \end{aligned}$$

We use the standard relation $\alpha^\mu_0 = \frac{1}{\sqrt{\alpha}}P^\mu$, with the dimensionful parameter α related to the conventional Regge slope α' by $\alpha = \frac{1}{2\alpha'}$. For later convenience we introduce the transverse Virasoro generators

$$L^{\rm T}_n = \tfrac{1}{2}\sum_{m=-\infty}^{+\infty} :\overline{\alpha}_{-m}\cdot\overline{\alpha}_{n+m}: \quad , \tag{2}$$

satisfying the algebra:

$$[L^{\rm T}_n, L^{\rm T}_m] = (n-m)L^{\rm T}_{n+m} + \tfrac{D-2}{12}(n^3-n)\delta_{n,-m} \quad .$$

In the non-critical light-cone string model longitudinal excitations are described by the Verma module $\mathcal{V}^{\rm L}(b)$ generated by the Virasoro algebra

$$[L^{\rm L}_n, L^{\rm L}_m] = (n-m)L^{\rm L}_{n+m} + \tfrac{c}{12}(n^3-n)\delta_{m,-n} \tag{3}$$

out of the highest wight state

$$L_0^{\rm L}\,\Omega^{\rm L}(b) = b\,\Omega(b) \quad , \quad L_n^{\rm L}\,\Omega^{\rm L}(b) \;=\; 0 \;\; , \;\; n>0 \quad .$$

For the central charge c of this algebra in the range $1 < c < 25$ and for $b > 0$ the hermicity properties of the generators:

$$L_n^{{\rm L}\,\dagger} = L_{-n}^{\rm L} \;\; , \;\; n \in \mathbb{Z} \quad ,$$

determine a positively defined non-degenerate inner product inducing a Hilbert space structure on $\mathcal{V}^{\rm L}(b)$. For $b = 0$ this inner product acquires null directions and for $b < 0$ one gets negative norm states in $\mathcal{V}^{\rm L}(b)$.

The full space of states in the noncritical light-cone string model is defined as the direct integral of Hilbert spaces

$$H_{\rm lc} = \int_{\mathbb{R}\backslash\{0\}} \frac{dp_+}{|p^+|} \int_{\mathbb{R}^{d-2}} d^{d-2}\overline{p}\;\mathcal{F}^{\rm T}(p^+,\overline{p}) \otimes \mathcal{V}^{\rm L}(b) \quad .$$

In order to complete the construction one has to introduce a unitary realization of the Poincare algebra on $H_{\rm lc}$. The generators of translations in the longitudinal and the transverse directions are given by the operators P^+ and $P^i, i = 1, ..., d{-}1$, respectively. The generator of translation in the x^+-direction is defined by the self-adjoint operator

$$P^- = \frac{\alpha}{P^+}(L_0^{\rm T} + L_0^{\rm L} - a_0) \quad . \tag{4}$$

Within the light-cone formulation the x^+ coordinate is regarded as an evolution parameter. In consequence P^- plays the role of the Hamiltonian and the Schrödinger equation reads

$$i\frac{\partial}{\partial x^+}\Psi = P^-\Psi \quad .$$

The generators of Lorentz rotations are defined by the self-adjoint operators

$$\begin{aligned}
M^{ij}_{\rm lc} &= P^i x^j - P^j x^i + i\sum_{n\geq 1}\frac{1}{n}(\alpha^i_{-n}\alpha^j_n - \alpha^j_{-n}\alpha^i_n) \quad , \\
M^{i+}_{\rm lc} &= P^+ x^i \quad , \\
M^{+-}_{\rm lc} &= \tfrac{1}{2}(P^+ x^- + x^- P^+) \quad , \\
M^{i-}_{\rm lc} &= \tfrac{1}{2}(x^i P^- + P^- x^i) - P^i x^- \\
&- \; i\frac{\sqrt{\alpha}}{P^+}\sum_{n\geq 1}\frac{1}{n}\left(\alpha^i_{-n}(L^{\rm T}_n + L^{\rm L}_n) - (L^{\rm T}_{-n} + L^{\rm L}_{-n})\alpha^i_n\right),
\end{aligned} \tag{5}$$

The algebra of the generators P^+, P^i, P^- (4), and $M^{\mu\nu}$ (5) closes to the Lie algebra of Poincare group if and only if the central charge c of the Virasoro algebra generating the "longitudinal" Verma module $\mathcal{V}^{\rm L}(b)$ and a_0 entering the definition of the Hamiltonian of the system (4) take the critical values

$$c = 26 - D \quad , \quad a_0 = 1 \quad .$$

Indeed, tedious but straightforward calculations show that the only anomalous terms appear in the commutators:

$$\begin{aligned}
\left[\, M^{i-}_{\rm lc} \;,\; M^{j-}_{\rm lc} \right] \;&=\; -\left(2 - \tfrac{D-2+c}{12}\right)\frac{\alpha}{P^{+2}}\sum_{n>0} n(\alpha^i_{-n}\alpha^j_n - \alpha^j_{-n}\alpha^i_n) \\
&\quad -\left(\tfrac{D-2+c}{12} - 2a_0\right)\frac{\alpha}{P^{+2}}\sum_{n>0}\frac{1}{n}(\alpha^i_{-n}\alpha^j_n - \alpha^j_{-n}\alpha^i_n) \quad .
\end{aligned}$$

Let us note that the model still contains one free parameter b entering the mass shell condition

$$M^2 = 2P^+P^- - \overline{P}^2 \;=\; 2\alpha\,(R_{\rm lc} + b - 1) \quad , \tag{6}$$

where

$$R_{\rm lc} = L_0^{\rm T} - \tfrac{1}{2}\overline{\alpha}_0^2 + L_0^{\rm L} - b \tag{7}$$

is the light-cone level operator. The only restriction on b is $b \geq 0$. As it was mentioned above this is a necessary and sufficient condition for the absence of ghost states in the "longitudinal" Verma module with the central charge $c = 26 - D$.

In order to analyse the spin content of the model it is convenient to work with the explicit Fock space realization [10, 19] of the "longitudinal" Verma module which can be constructed for $b \geq \frac{25-D}{24}$. Let us denote by $\mathcal{F}^{\rm L}(q)$ the Fock space generated by the infinite algebra of "Liouville" excitation operators:

$$[\beta_m, \beta_n] = m\delta_{m,-n} \quad , \quad (\beta_m)^\dagger \;=\; \beta_{-m} \quad , \quad m, n \in \mathbb{Z} \quad ,$$

out of the unique vacuum state $\Omega(q)$ satisfying

$$\beta_0\Omega(q) \;=\; q\Omega(q) \;\; , \;\; \beta_m\Omega(q) \;=\; 0 \;\; , \;\; m > 0 \quad .$$

For each $n \in \mathbb{Z}$ and a dimensionless real parameter β we define

$$L_n^{\rm L} = \tfrac{1}{2}\sum_{k=-\infty}^{+\infty} :\beta_{-k}\beta_{n+k}: + 2\sqrt{\beta}\, i n\beta_n + 2\beta\delta_{n,0} \quad , \tag{8}$$

satisfying

$$\begin{aligned} [L_n^{\rm L}, L_m^{\rm L}] &= (n-m)L_{n+m}^{\rm L} + \tfrac{1+48\beta}{12}(n^3 - n)\delta_{n,-m} \quad , \\ L_n^{{\rm L}\,\dagger} &= L_{-n}^{\rm L} \quad , \\ L_0^{\rm L}\Omega(q) &= (\tfrac{1}{2}q^2 + 2\beta)\Omega(q) \quad . \end{aligned}$$

It follows that for $\beta = \frac{25-D}{48}$, and $q = \sqrt{2b - \frac{25-D}{12}}$ the operators (8) can be identified with the Virasoro generators of the "longitudinal" Verma module (3) and the spaces $\mathcal{F}^{\rm L}(q)$, and $\mathcal{V}^{\rm L}(b)$ are isomorphic with respect to the inner product structure.

The generators of the Poincare algebra regarded as operators on the Fock space

$$H_{\rm lc} = \int_{\mathbb{R}\setminus\{0\}} \frac{dp_+}{|p^+|} \int_{\mathbb{R}^{d-2}} d^{d-2}\overline{p}\; \mathcal{F}^{\rm T}(p^+, \overline{p}) \otimes \mathcal{F}^{\rm L}(q) \quad , \tag{9}$$

are uniquely determined by their action on the vacuum states $\Omega(p^+, \overline{p}) \otimes \Omega(q)$ and the commutation relations with the excitation operators:

$$\begin{aligned} \left[P^-, \alpha_n^i\right] &= -\alpha\frac{n}{p^+}\alpha_n^i \quad , \quad \left[P^-, \beta_n\right] \;=\; -\alpha\frac{n}{p^+}\beta_n \quad , \\ \left[M_{\rm lc}^{ij}\,, \alpha_n^k\right] &= -i(\alpha_n^j\delta^{ik} - \alpha_m^i\delta^{jk}) \qquad , \\ \left[M_{\rm lc}^{i-}, \beta_n\right] &= -\alpha\frac{nx^i}{p^+}\beta_n + \sqrt{\alpha}\frac{in}{p^+}\left(\sum_{m>0}\frac{1}{m}(\alpha_{-m}^i\beta_{n+m} - \alpha_m^i\beta_{n-m})\right) \\ &\quad -\sqrt{\alpha}\frac{2}{p^+}n\sqrt{\beta}\alpha_n^i \quad , \\ \left[M_{\rm lc}^{i-}, \alpha_n^j\right] &= -\alpha\frac{nx^i}{p^+}\alpha_n^j + \sqrt{\alpha}\frac{in}{p^+}\left(\sum_{m>0}\frac{1}{m}(\alpha_{-m}^i\alpha_{m+n}^j - \alpha_{n-m}^j\alpha_m^i)\right) \\ &\quad +\sqrt{\alpha}\frac{i}{p^+}\delta^{ij}\,(L_n^{\rm T} + L_n^{\rm L}) \quad , \end{aligned} \tag{10}$$

with all remaining commutators being zero.

2 Little Group SO$(D-1)$ and Spin Content

In this section we shall analyse the spin content of the noncritical light-cone string using the Fock space realization. Our results are therefore limited to the models with the parameter b in the range $b \geq \frac{25-D}{24}$. Since the generators of the Poincare algebra (4,5) commute with the light-cone level operator $R_{\rm lc}$ (7) entering the mass shell condition (6) the decomposition of $H_{\rm lc}$ into representations of a fixed mass coincides with the level decomposition

$$H_{\rm lc} = \bigoplus_{N\geq 0} H_{\rm lc}^{(N)} \quad , \quad R_{\rm lc}\, H_{\rm lc}^{(N)} \;=\; N H_{\rm lc}^{(N)} \quad .$$

For b in the range $\frac{25-D}{24} \leq b < 1$ the lowest level subspace $H_{\rm lc}^{(0)}$ carries the irreducible tachyonic representation with $m^2 = b-1$. For $b=1$ $H_{\rm lc}^{(0)}$ is a massless, and for $b>1$ a massive scalar representation. This completes analysis of the spin content of the lowest level.

For the whole range $\frac{25-D}{24} < b$ all higher level representations $H_{\rm lc}^{(N)}$ have real non-zero mass and can be further decomposed into particle $H_{\rm lc}^{+(N)}$ and antiparticle $H_{\rm lc}^{-(N)}$ representations. We shall find the decomposition of $H_{\rm lc}^{+(N)}$ into irreducible representations.

Let us fix some level $N>0$. For any momentum p satisfying $-p^2 = m^2 = 2\alpha(N+b-1)$ and $p^+>0$ one can always choose a light-cone frame $\{k, k', e^i\}$ such that

$$p = \sqrt{\alpha} k' + \sqrt{\alpha}\,(N+b-1)\,k \quad .$$

In this frame the operators G^j, $j = 1...D-2$:

$$G^j = i\tfrac{m^2}{2\alpha} M^{j+} - iM^{j-} = -\sum_{n>0} \frac{1}{n}\left(\alpha^j_{-n}(L^{\rm T}_n + L^{\rm L}_n) - (L^{\rm T}_{-n} + L^{\rm L}_{-n})\alpha^j_n\right) \quad , \tag{11}$$

together with the infinitesimal transverse rotations of SO$(D-2)$

$$iM^{ji} = -\sum_{n>0} \frac{1}{n}(\alpha^j_{-n}\alpha^i_n - \alpha^i_{-n}\alpha^j_n) \quad , \tag{12}$$

generate unitary action of the little group SO$(D-1)$ of the momentum p.

One should notice that split of $so(D-1)$ onto (11) and (12) is of symmetric type i.e. (11) generate the whole Lie algebra by commutators and much of the analysis can be restricted to their representation only.

Let us consider the subspace $H_p \subset H_{\rm lc}^{+(N)}$ of all states of level N with the fixed momentum p $(p^+>0)$. We shall calculate the character

$$\chi_N(g) = {\rm tr}\mathcal{D}(g) \tag{13}$$

of the corresponding unitary representation $\mathcal{D}$ of the little group SO$(D-1)$ on H_p. For this purpose it is convenient to decompose H_p according to the partitions of N. We denote by $\mathcal{P}(N)$ the set of all partitions of N. Any partition $p(N) \in \mathcal{P}(N)$ is uniquely characterized by the sequence $p(N) = (m_N, m_{N-1}, ..., m_1)$ with $\sum k m_k = N$. For every partition $p(N)$ of N we define the corresponding Hilbert subspace $H_p^{p(N)} \subset H_p$ spanned by all states of the form

$$A^{a^{m_N}_N}_{-N} A^{a^1_{N-1}}_{-N+1} ... A^{a^{m_{N-1}}_{N-1}}_{-N+1} ... A^{a^1_1}_{-1} ... A^{a^{m_1}_1}_{-1}\, \Omega(p^+, \overline{p}) \otimes \Omega(q)$$

where, for $m > 0$

$$A^a_{-m} = \begin{cases} i\beta_{-m} & \text{for } a = 0 \ , \\ \alpha^a_{-m} & \text{for } a = 1, ..., d-2 \ , \end{cases}$$

and $q = \sqrt{2b - \frac{25-D}{12}}$. [1] Let V_{-m} be the vector space spanned by the the excitation operators $\{A^a_{-m}\}^{d-2}_{a=0}$ of level m. Then

$$H^{p(N)}_p \simeq (\otimes^{m_N}_S V_{-N}) \otimes ...... \otimes (\otimes^{m_1}_S V_{-1}) \quad , \tag{14}$$

where $\otimes^m_S$ denotes the m-th symmetric tensor power. The Hilbert space H_p decomposes into orthogonal sum

$$H_p = \bigoplus_{p(N)\in\mathcal{P}(N)} H^{p(N)}_p \tag{15}$$

and one has the corresponding partition of identity

$$id = \sum_{p(N)\in\mathcal{P}(N)} \pi_{p(N)} \tag{16}$$

with $\{\pi_{p(N)}\}$ being the set of projectors corresponding to (15). Inserting (16) on both sides of $\mathcal{D}(g)$ in (13) and using $\mathrm{tr}(\pi_{p(N)}\mathcal{D}(g)\pi_{p'(N)}) = 0$ for $p(N) \neq p'(N)$ one gets immediately that

$$\chi_N(g) = \sum_{p(N)\in\mathcal{P}(N)} \mathrm{tr}(\pi_{p(N)}\mathcal{D}(g)\pi_{p(N)}) \quad . \tag{17}$$

The virtue of the formula above is that the traces on the r.h.s. are characters of the representations of SO$(D-1)$ on the subspaces (14), regarded as appropriate tensor products of the fundamental vector representations on the $(D-1)$-dimensional vector spaces $V_{-m}, m > 0$. This is evident on the Lie algebra level. Infinitesimal action of $\mathcal{D}(g)$ on H_p (14) with g generated by (11) is given by the sum of commutators of (11) with the fundamental excitation operators:

$$\left[\, G^i \ , \ A^a_{-m} \right] \ = \ 2m\sqrt{\beta} D^{(i)\ a}_{\ \ b} A^b_{-m} + M^{(i)a}(m) \quad , \tag{18}$$

where $D^{(i)}$ are the standard antisymmetric matrices of fundamental representation of $so(D-1)$ with 1 on i-th place in the first row and zero elsewhere. The exact form of the operators $M^{(i)a}(m)$ can be easily obtained from (10). One should only take into account that in the kinematical situation assumed at the beginning of this section they are of second order in the excitation operators. Their contributions to the infinitesimal actions of SO$(D-1)$ on $H^{p(N)}_p$ correspond therefore to different partitions of N and are zero under projection $\pi_{p(N)}$. Consequently the operators $\pi_{p(N)}\mathcal{D}(g)\pi_{p(N)}$ define corresponding tensor representations of SO$(D-1)$ on the spaces $H^{p(N)}_p$. On the infinitesimal level they are described by the first term of the r.h.s. of (18).

Taking into account the above property of $\pi_{p(N)}\mathcal{D}(g)\pi_{p(N)}$ one can rewrite the formula (17) for χ_N in the following form

$$\chi_N = \sum_{p(N)\in\mathcal{P}(N)} \prod_{m_k\in p(N)} \chi^{m_k}_S \quad , \tag{19}$$

with $\chi^{m_k}_S$ being the character of m_k-th symmetric tensor power of the fundamental (vector) representation of SO$(D-1)$.

[1] The factor of i in front of β_{-m} is inserted in order to obtain the standard real representation of the so$(D-1)$ Lie algebra in (18).

In order to describe the spin content of the model it is convenient to introduce the "generating" function for characters:

$$\chi(t,g) := \sum_{N\geq 0} t^N \chi_N(g) \quad . \tag{20}$$

Inserting (19) into (20) and taking into account the formula for symmetric characters [20] one obtains:

$$\chi(t,g) \;=\; \prod_{k\geq 1} \frac{1}{\det(1 - t^k \mathcal{D}_f(g))} \quad , \tag{21}$$

where $\mathcal{D}_f$ denotes fundamental representation of $\mathrm{SO}(D-1)$. As the characters are class functions the domain of (21) can be restricted to the maximal torus of $\mathrm{SO}(D-1)$ and in fact to the fundamental domain describing the quotient $T_{\mathrm{SO}(D-1)}/\mathcal{W}$ of the torus by the Weyl group.

The method of finding character generating functions for critical bosonic and fermionic strings was presented in details in the paper [21]. The formulae obtained in that article after multiplication by the partition function $p(t) = \prod_{n=1}^{\infty}(1-t^n)^{-1}$ give the results for massive string. This extra factor in front of generating functions corresponds to the fact that compared to the critical models there is one additional family of modes to generate massive string states.

We give the formulae for the character generating function in most interesting case of $D=4$. This case is also technically simplest as the maximal torus of the little group SO(3) is 1-dimensional and the representations are labelled by one non-negative integer. The function (21) can be written as:

$$\chi(t,\varphi) = p(t)^3 \sum_{k>0}\sum_{j\geq 0} t^{\frac{k(k-1)}{2}} (-1)^{k-1} (1-t^k)^2 t^{kj} \chi^j(\varphi) \quad , \tag{22}$$

where

$$\chi^j(\varphi) = \frac{\sin\left((j+\frac{1}{2})\varphi\right)}{\sin\left(\frac{\varphi}{2}\right)} \quad , \tag{23}$$

is the irreducible character of spin $j \in I\!N$. The spin spectrum of the noncritical light-cone string with the physical intercept

$$\alpha(0) = b - 1 \;=\; \tfrac{1}{2}q^2 - \tfrac{D-1}{24} \;=\; -\tfrac{D-1}{24}$$

up to the 12th excited level is presented on Fig.1. The first 4 levels of the corresponding closed string model are also displayed for comparison.

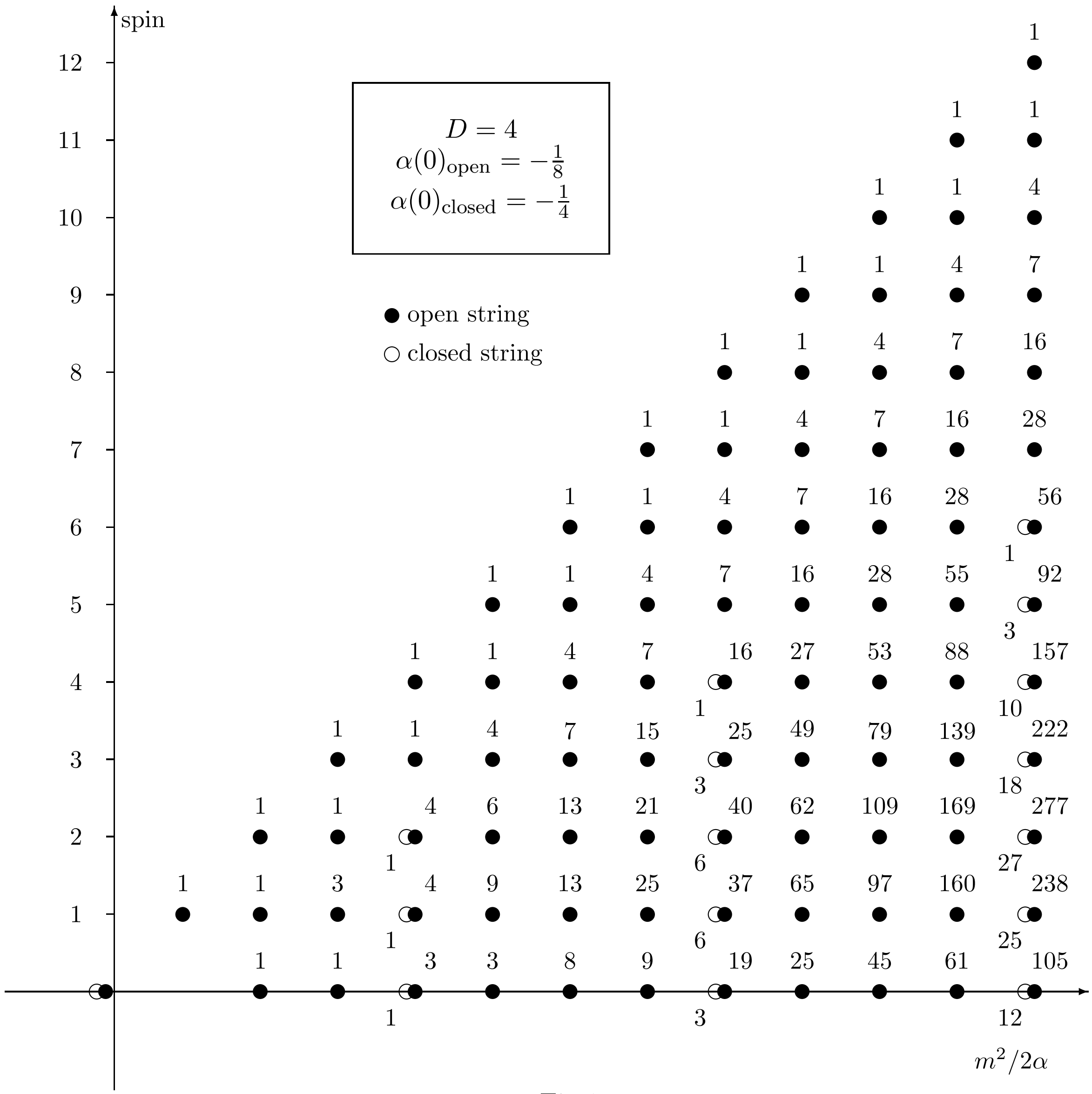

Fig.1

3 Closed Noncritical Light-Cone String

Follwoin standard lines the construction of Section 2 can be extended to the closed string case following standard lines. The closed string Hilbert space is the "diagonal" part of the tensor product

$$H_{\rm cl} = \widetilde{H}_{\rm lc} \otimes_{\rm D} H_{\rm lc} \;=\; \bigoplus_{N\geq 0} \widetilde{H}^{(N)}_{\rm lc} \otimes H^{(N)}_{\rm lc} \quad ,$$

of two copies of the open string Hilbert spaces. The canonical variables in the sector $H_{\rm lc}$ are usually called right-movers while those in the sector $\widetilde{H}_{\rm lc}$ - left movers. The right and

the left-movers commute with each other by construction. Both sectors are related by the conditions:

$$
\begin{aligned}
\tilde{\alpha}^{\mu}_0 &= \alpha^{\mu}_0 = \frac{P^{\mu}}{2\sqrt{\alpha}} \quad , \quad \mu = 0, ..., D-1 \quad , \\
\widetilde{c} &= c \quad , \quad \widetilde{b} = b \quad .
\end{aligned}
$$

The Hamiltonian generating the x^+-evolution of the system is given by

$$
P^-_{\rm cl} = \frac{\alpha}{P^+}(\widetilde{L}^{\rm T}_0 + L^{\rm T}_0 + \widetilde{L}^{\rm L}_0 + L^{\rm L}_0 - 2a_0) \quad . \tag{24}
$$

Similarly the other Poincare generators are defined by the formulae

$$
\begin{aligned}
M^{ij}_{\rm lc} &= P^i x^j - P^j x^i \\
&\quad + i \sum_{n\geq 1} \frac{1}{n} (\tilde{\alpha}^i_{-n}\tilde{\alpha}^j_n - \tilde{\alpha}^j_{-n}\tilde{\alpha}^i_n + \alpha^i_{-n}\alpha^j_n - \alpha^j_{-n}\alpha^i_n) \quad , \\
M^{i+}_{\rm lc} &= P^+ x^i \quad , \\
M^{+-}_{\rm lc} &= \tfrac{1}{2}(P^+ x^- + x^- P^+) \quad , \\
M^{i-}_{\rm lc} &= \tfrac{1}{2}(x^i P^-_{\rm cl} + P^-_{\rm cl} x^i) - P^i x^- \\
&\quad - i\frac{\sqrt{\alpha}}{P^+} \sum_{n\geq 1} \frac{1}{n} \Big(\tilde{\alpha}^i_{-n}(\widetilde{L}^{\rm T}_n + \widetilde{L}^{\rm L}_n) - (\widetilde{L}^{\rm T}_{-n} + \widetilde{L}^{\rm L}_{-n})\tilde{\alpha}^i_n \\
&\qquad\qquad + \alpha^i_{-n}(L^{\rm T}_n + L^{\rm L}_n) - (L^{\rm T}_{-n} + L^{\rm L}_{-n})\alpha^i_n \Big) \quad ,
\end{aligned} \tag{25}
$$

The conditions for the closure of the Poincare algebra are $a_0 = 1$ and the central charges in the right and the left "longitudinal" Verma module satisfying $\widetilde{c} = c = 26 - D$.

As in the case of the open string the longitudinal Verma module can be realized in a Fock space for $b > \frac{25-D}{24}$. The left and the right "Liouville" sectors are related by the condition $\widetilde{q} = q$. The mass shell condition reads

$$
M^2 = 2P^+P^- - \overline{P}^2 \; = \; 4\alpha\,(2N + \alpha(0)) \quad ,
$$

with the "physical" intercept given by $\alpha(0) = 2b - 2 = q^2 - \frac{D-1}{12}$.

The generating function for characters can be identified with the "diagonal" part (i.e. all terms of the form $x^N y^N$) of the product of two open string generating functions. In the case $D = 4$ one has

$$
\begin{aligned}
\chi(x, y, \varphi) &= \chi(x, \varphi)\chi(y, \varphi) \\
&= p^3(x)p^3(y) \sum_{k=1}^{\infty}\sum_{i=0}^{\infty}\sum_{l=1}^{\infty}\sum_{j=0}^{\infty}\sum_{m=|i-j|}^{i+j} \\
&\quad (-1)^{k+l} x^{\frac{k(k-1)}{2}+ki} y^{\frac{l(l-1)}{2}+lj} (1-x^k)^2(1-y^l)^2 \frac{\sin\left((m+\frac{1}{2})\varphi\right)}{\sin\left(\frac{\varphi}{2}\right)} \quad .
\end{aligned}
$$

The spectrum of the closed light-cone string in 4 dimensions and with the "physical" intercept $\alpha(0) = -\frac{1}{4}$ is presented for first 11 excited levels and up to the spin 11 on Fig.2.

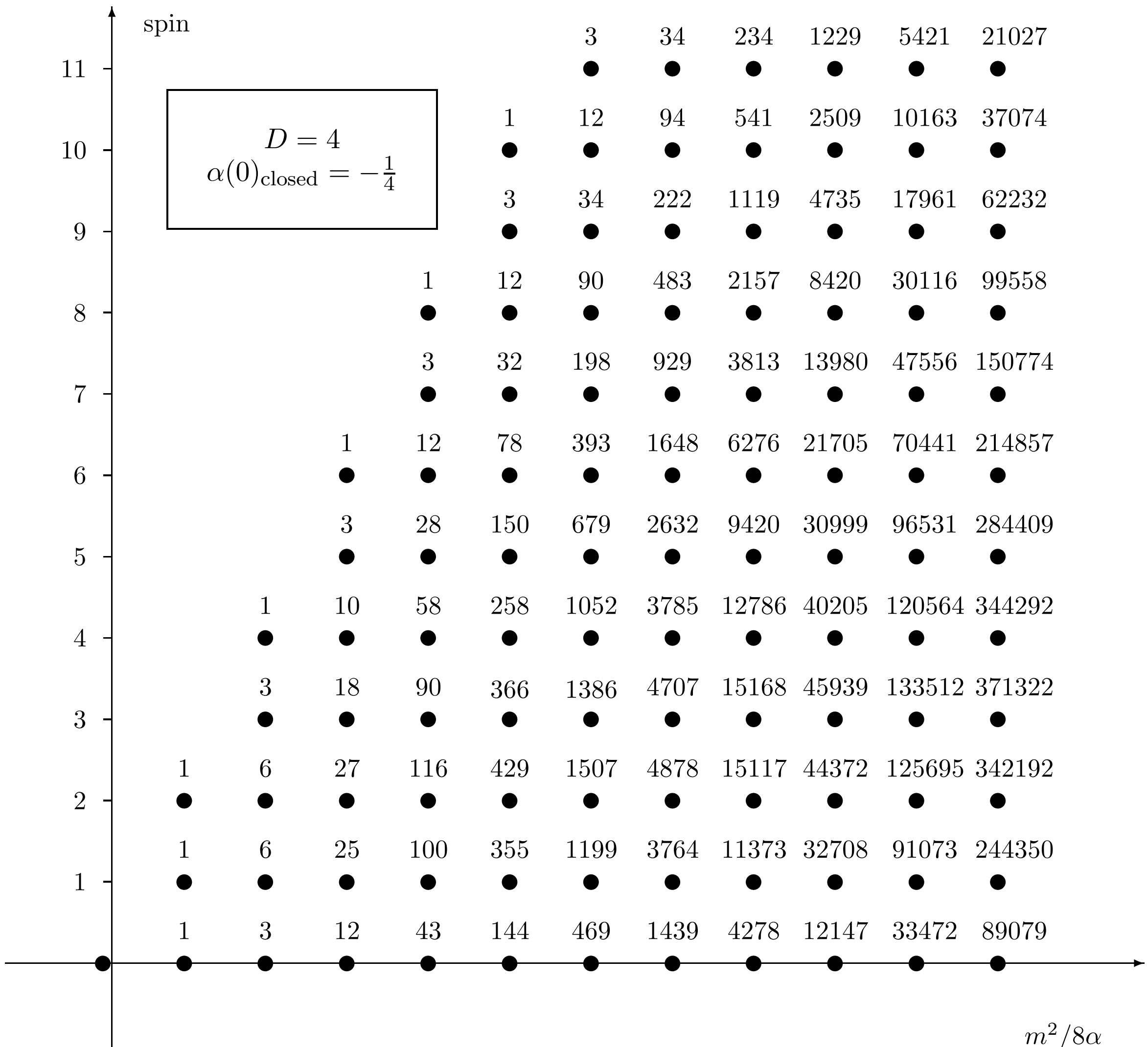

Fig.2

4 Critical massive string

The quantum massive string model is formulated in terms of the pseudo-Hilbert space defined [17] as a direct integral of Fock spaces H_p:

$$H = \int d^D p \; H_p \quad . \tag{26}$$

The integration ranges over D-dimensional spectrum of the self-adjoint momentum operators $\{P^\mu; \mu = 0, .., D-1\}$, which together with their canonical conjugates satisfy standard commutation relations:

$$[P^\mu, x^\nu] \;\; = \;\; -i\eta^{\mu\nu}$$

Every space H_p is generated by the infinite algebra of the excitation operators:

$$[\alpha^\mu_m, \alpha^\nu_n] \;\; = \;\; m\eta^{\mu\nu}\delta_{m,-n} \quad ,$$

$$[\beta_m, \beta_n] \;=\; m\delta_{m,-n}; \quad , \quad m,n \in \mathbb{Z} \quad , \tag{27}$$

out of the unique vacuum state Ω_p satisfying

$$\begin{aligned} \alpha^\mu_m \Omega_p &= \beta_m \Omega_p \;=\; 0 \quad , \quad m > 0 \quad , \\ \alpha^\mu_0 \Omega_p &= \frac{p^\mu}{\sqrt{\alpha}} \Omega_p \quad , \quad \beta_0 \Omega_p \;=\; q\Omega_p \quad , \end{aligned}$$

where the convention $\alpha^\mu_0 := \frac{1}{\sqrt{\alpha}} P^\mu$ is used. The eingenvalue q of the operator β_0 is regarded as a free parameter of the construction and is the same for all Ω_p. The scalar product in H_p is induced by (27) and the conjugation properties

$$(\alpha^\mu_m)^\dagger = \alpha^\mu_{-m} \; , \; (\beta_m)^\dagger = \beta_{-m} \quad .$$

There is a realization of the Poincare algebra on H with the generators of translations and Lorentz rotations given by:

$$P^\mu \quad , \quad \text{and} \quad M^{\mu\nu} = P^\mu x^\nu - P^\nu x^\mu + i \sum_{n=1}^{+\infty} \frac{1}{n} (\alpha^\mu_{-n} \alpha^\nu_n - \alpha^\nu_{-n} \alpha^\mu_n) \quad , \tag{28}$$

respectively.

One introduces the infinite set of quantum constraints

$$L_n = \frac{1}{2} \sum_{m=-\infty}^{+\infty} : \alpha_{-m} \cdot \alpha_{n+m} : + \frac{1}{2} \sum_{m=-\infty}^{+\infty} : \beta_{-m} \beta_{n+m} : + 2\sqrt{\beta} i n \beta_n + 2\beta \delta_{n,0} \quad , \tag{29}$$

satisfying the Virasoro algebra:

$$[L_n, L_m] = (n-m) L_{n+m} + \tfrac{D+1+48\beta}{12} (n^3 - n) \delta_{n,-m} \quad .$$

The space of physical states is defined as

$$\mathcal{H} = \{\Psi \in H : \; (L_n - \delta_{n,0} a_0) \Psi = 0 \; , n \geq 0\} \quad .$$

The L_0 operator is a combination of the momentum and level operator R and the corresponding constraint yields the mass shell condition:

$$(L_0 - a_0) \, \Psi \;=\; \left(\tfrac{1}{2\alpha} P^2 + R - \alpha(0) \right) \Psi \;=\; 0 \quad , \tag{30}$$

where $\alpha(0) = a_0 - \frac{1}{2} q^2 - 2\beta$ is the physical intercept of leading Regge trajectory.

In the massive string model [16, 17] one has an extra constraint $\beta_0 \Psi = 0$. In the present paper we relax this condition keeping the arbitrary real eigenvalue q of the operator β_0 as a free parameter. In contrast to the parameters β and a_0 which are restricted by the no-ghost theorem q may take arbitrary real value.

Since the Poincare generators (28) commute with the constraints (29) they define a representation of the Poincare algebra on $\mathcal{H}$. It follows from the mass shell condition (30) that the decomposition of $\mathcal{H}$ into representations of a fixed mass coincides with the level structure:

$$\mathcal{H} \;=\; \bigoplus_{N \geq 0} \mathcal{H}^{(N)} \quad ; \quad R \, \mathcal{H}^{(N)} = N \mathcal{H}^{(N)} \quad .$$

Each subspace $\mathcal{H}^{(N)}$ is further decomposed into a direct integral of finite dimensional spaces $\mathcal{H}^{(N)}(p)$ with fixed on-shell momentum:

$$\mathcal{H}^{(N)} = \int_{\mathcal{S}_N} d\mu^N(p)\, \mathcal{H}^{(N)}(p) \quad ,$$

where $\mathcal{S}_N$ denotes the mass-shell at level N determined by the condition (30) $-p^2 = m^2 = 2\alpha(N + \frac{1}{2}q^2 + 2\beta - a_0)$, and $d\mu^N(p)$ denotes the Lorentz invariant measure on $\mathcal{S}_N$.

In order to analyse the physical content of the model one needs a tractable parameterization of the space $\mathcal{H}$ of physical states. A global one can be obtained by an appropriate modification [16], [17] of the standard DDF operators [18], [3]. The DDF construction starts with fixing a light-cone frame $\{k, k', e^1, ..., e^{D-2}\}$. The transverse $D-2$ coordinates $p^i = e^i \cdot p$, $i = 1, ..., D-2$, and the light-cone coordinate $p^+ = k \cdot p$ provide a global regular parameterization of all mass shells corresponding to positive mass square and a singular one in the case of tachyon. The light-cone coordinate $p^- = k' \cdot p$ on $\mathcal{S}_N$ becomes dependent and equals to $p^- = \frac{1}{2p^+}(\overline{p}^2 + 2\alpha(N - \alpha(0))$. Since the points with $p^+ = 0$ form a zero-measure subset of the tachyonic mass shell this singularity of light-cone coordinates has no effect in the quantum theory.

The construction of an appropriate set of the DDF operators for a given light-cone basis is briefly presented in the appendix A 1. Let us only remark that the definitions (A.1), (A.2), (A.3) differ from the standard ones by the replacement

$$k \longrightarrow \frac{\sqrt{\alpha}}{k \cdot P} k \quad , \quad k' \longrightarrow \frac{k \cdot P}{\sqrt{\alpha}} k' \quad .$$

Due to this slight modification the DDF operators are well defined on the whole space $\mathcal{H}$, while in the conventional constructions the domain is restricted by the condition $\frac{k \cdot P}{\sqrt{\alpha}} = k \cdot \alpha_0 \in \mathbb{Z}$.

There are D families of DDF operators creating the physical states: $D-2$ families $\{A^i_m\}_{m \in \mathbb{Z}}$ generating transverse excitations (A.1), one family $\{C_m\}_{m \in \mathbb{Z}}$ generating Liouville excitations (A.3), and one family $\{B_m\}_{m \in \mathbb{Z}}$ of shifted Brower operators (A.6) corresponding to the longitudinal ones. These operators satisfy diagonalized algebra

$$\begin{array}{rcl} [A^i_m, A^j_n] & = & m\delta^{ij}\delta_{m,-n} \quad , \\ [C_m, C_n] & = & m\delta_{m,-n} \quad , \\ [B_n, B_m] & = & (n-m)B_{n+m} + \frac{25-D-48\beta}{12}(n^3 - n)\delta_{n,-m} \quad . \end{array} \tag{31}$$

with all remaining commutators being zero. It is convenient to introduce additional family $\{F_m\}_{m \in \mathbb{Z}}$ of operators (A.4) satisfying the commutation relations [22, 17]

$$\begin{array}{rcl} [A^i_m, F_n] & = & [C_m, F_n] \; = \; [F_m, F_n] \; = \; 0 \quad , \\ [B_m, F_n] & = & -nF_{n+m} \quad , \end{array}$$

In contrast to the DDF operators F_m do not commute with the constraints (29). All the DDF and F operators have definite level with respect to the original covariant level operator (7):

$$[\, R \, , \; D_m] \; = \; -mD_m,$$

where D denotes any of A^i, B, C, F. The virtue of introducing F operators is that for every $N \geq 1$ and every $p \in \mathcal{S}_N$ with $p^+ \neq 0$, all ordered monomials of the DDF and F operators of level N

$$\vartheta^N(A^i, B, C, F)\Omega_{p + \frac{\sqrt{\alpha}N}{k \cdot p}k} \tag{32}$$

form a basis in the subspace $H^{(N)}(p) \subset H$ of all states from H with the on-shell momentum p [22, 17]. It follows in particular that the states generated only by the DDF operators exhaust all physical states.

Let us note that the condition $p^+ = 0$ can be satisfied only by the tachyonic on-shell momentum, but even in this case momenta with $p^+ = 0$ form a zero-measure subset of the mass shell. Hence the modified DDF construction provides a global parameterization of the space $\mathcal{H}$.

In the DDF parametrization of the space of physical states the inner product structure of $\mathcal{H}$ is completely determined by the B-sector of the algebra (18). The general discussion of the "no-ghost" theorem was given in [17]. Here we are only interested in the critical model in which the subspace of null states $\mathcal{H}_0 = \{\Psi \in \mathcal{H} \ : (\ \Psi, \cdot\ ) = 0\}$ is largest possible. As it can be easily inferred from the construction of the shifted Brower longitudinal operator B_0 (A.6) and the algebra (31) this is the case of the critical values of parameters $\beta = \frac{25-D}{24}, a_0 = 1$. For these values all states containing B-excitations are null.

Since physical states which differ by a null state carry the same physical information the space of "true" physical states is given by the quotient $\mathcal{H}_{\rm ph} = \mathcal{H}/\mathcal{H}_0$. The representation (28) of the Poincare algebra on $\mathcal{H}$ induces a representation on $\mathcal{H}_{\rm ph}$. For each Poincare generator M on $\mathcal{H}$ one defines the corresponding generator $M_{\rm ph}$ by its action on equivalence classes $M_{\rm ph}[\Psi] = [M\Psi]$, $[\Psi] \in \mathcal{H}_{\rm ph}$.The physical content of the quantum model is completely determined by the decomposition of $\mathcal{H}_{\rm ph}$ into unitary irreducible representations.

5 Light-Cone Gauge

As it was mentioned in the introduction one way to parameterize the quotient $\mathcal{H}_{\rm ph} = \mathcal{H}/\mathcal{H}_0$ is to consider a subspace $\mathcal{H}_{\rm gauge} \subset \mathcal{H}$ containing only one element from each equivalence class in $\mathcal{H}_{\rm ph}$. Such a subspace can be regarded as a gauge slice for the quantum symmetry acting on states in $\mathcal{H}$ by shifts in the null direction $\mathcal{H}_0$.

In this section we shall consider the light-cone gauge which is defined as the subspace $\mathcal{H}_{\rm LC} \subset \mathcal{H}$ of all physical states generated by the the transverse A^i (A.1), and the Liouville C (A.3) DDF operators. Since in the critical massive string model all DDF states containing shifted Brower modes B (A.6) are null, the subspace $\mathcal{H}_{\rm LC}$ is a good gauge slice. It defines a section $\Sigma : \mathcal{H}_{\rm ph} \to \mathcal{H}$ of the fibration $\mathcal{H} \to \mathcal{H}/\mathcal{H}_0 = \mathcal{H}_{\rm ph}$. Σ is an isomorphism of Fock spaces. The generators of the representation of the Poincare algebra induced by Σ on $\mathcal{H}_{\rm lc}$ can be expressed directly in terms of generators M (28) on $\mathcal{H}$

$$M_{\rm LC}\Psi = \pi_{\rm LC} M\Psi \quad , \tag{33}$$

where $\Psi \in \mathcal{H}_{\rm LC}$ and $\pi_{\rm LC}$ is the projection onto the light-cone gauge slice $\mathcal{H}_{\rm LC}$ defined in the base of diagonal DDF operators by neglecting all terms containing the shifted Brower B-excitations (A.6).

We shall show that the critical massive string is isomorphic with the noncritical light-cone string introduced in Section 1. Our strategy to prove this equivalence is to identify the space of states $H_{\rm lc}$ of the noncritical light-cone string as a subspace of the (pseudo) Hilbert space H defining the massive string model and to construct an isomorphism $\mathcal{H}_{\rm LC} \to H_{\rm lc}$ as a projection along null direction.

Let as introduce an auxiliary subspace $H_{\rm aux} \subset H$ defined by

$$H_{\rm aux} \quad = \quad \bigoplus_{N\geq 0} H^{(N)}_{\rm aux} \quad ,$$

$$H_{\rm aux}^{(N)} \;=\; \int_{\mathcal{S}_N} d\mu^N(p)\; H_{\rm aux}^{(N)}(p) \quad , \tag{34}$$

where $\mathcal{S}_N$ denotes the mass-shell at level N defined by the equation $m^2 = 2\alpha(N+\frac{1}{2}q^2-\frac{D-1}{24})$, and $H_{\rm aux}^{(N)}(p)$ is the subspace of all states generated out of the vacuum Ω_p by all polynomials of level N in the creation operators α^+, α^i, and β. The subspace $H_{\rm aux}$ has non-negative inner product with a large subspace of null states. All states containing α^+-excitations are null. If the "Liouville" momenta q are the same in both models one can identify the space of states $H_{\rm lc}$ of the noncritical light-cone string (9) as the subspace of $H_{\rm aux}$ containing all states generated only by α^i, and β operators. Let us stress that the states from $H_{\rm lc}$ are not physical states from the point of view of the massive string model, $H_{\rm lc}\cap\mathcal{H}=\{0\}$. The auxiliary space $H_{\rm aux}$ intersects the space of physical states $\mathcal{H}$ along the light-cone slice $\mathcal{H}_{\rm LC} = H_{\rm aux}\cap\mathcal{H}$. Indeed for states $\Psi\in\mathcal{H}_{\rm LC}^{(N)}(p)$ one gets

$$\begin{aligned}\Psi \;&=\; \vartheta^N(A^i,C)\;\Omega_{p+\frac{\sqrt{\alpha}N}{k\cdot p}k} \\ &=\; \vartheta^N(\alpha^i,\beta)\Omega_p + \vartheta^N_{\rm rest}(\alpha^+,\alpha^i,\beta)\Omega_p \;\in\; H_{\rm aux}^{(N)}(p) \quad ,\end{aligned} \tag{35}$$

where each term of the polynomial $\vartheta^N_{\rm rest}(\alpha^+,\alpha^i,\beta)$ contains at least one creation operator α^+.

The virtue of introducing the subspace $H_{\rm aux}$ is that it contains both $H_{\rm lc}$ and $\mathcal{H}_{\rm LC}$. Moreover one can obtain one subspace from the other by deformation in the null direction which considerably simplifies calculations of the induced Poincare generators.

Let $\pi_{\rm lc}: H_{\rm aux}\to H_{\rm lc}$ be the projection on the subspace $H_{\rm lc}$ defined by neglecting all terms containing α^+-excitations. It follows from (35) that

$$\sigma_{\rm lc}\;:\; H_{\rm lc}\ni\vartheta^N(\alpha^i,\beta)\Omega_p \longrightarrow \vartheta^N(A^i,C)\;\Omega_{p+\frac{\sqrt{\alpha}N}{k\cdot p}k}\in\mathcal{H}_{\rm LC} \quad . \tag{36}$$

is a section of $\pi_{\rm lc}$ ($\pi_{\rm lc}\circ\sigma_{\rm lc}={\rm id}$). The composition $\pi_{\rm lc}\circ\Sigma:\mathcal{H}_{\rm ph}\to H_{\rm lc}$ is an isomorphism of Hilbert spaces and induces on $H_{\rm lc}$ a representation of the Poincare algebra with the generators

$$\pi_{\rm lc}\circ\pi_{\rm LC}\circ M\circ\sigma_{\rm lc}$$

where M are generators (28) on $\mathcal{H}$. In order to compare this representation with the light-cone string representation introduced in Section 1 one has to calculate the generators in terms of the initial conditions with respect to the x^+-evolution.

Note that by construction all states in $H_{\rm aux}$ (34) have on-mass-shell momenta. In particular for states from $H_{\rm lc}^{(N)}(p)\subset H_{\rm aux}^{(N)}(p)$ the x^+-dependence is given by

$$\vartheta^N(\alpha^i,\beta)\Omega_p(p^+,\overline{p},x^+) = \vartheta^N(\alpha^i,\beta){\rm e}^{\frac{ix^+}{2p^+}(\overline{p}^2+2\alpha(N-\alpha(0)))}\Omega_p(p^+,\overline{p},0) \quad . \tag{37}$$

For every operator A on H we define the operator $\overline{A}$ acting on the space of initial conditions $\overline{\Psi}(p^+,\overline{p}) = \Psi(p^+,\overline{p},x^+)_{|x^+=0}$ of states from $H_{\rm lc}$ by

$$\overline{A}\,\overline{\Psi} = \overline{\pi_{\rm lc}\circ\pi_{\rm LC}\circ\pi_{\rm ph}\circ A\circ\sigma_{\rm lc}\Psi} \quad ,$$

where $\pi_{\rm ph}: H\to\mathcal{H}$ is the projection on the space of physical states defined in the base (32) by neglecting all terms containing the F operators. For the generators of translation one gets

$$\overline{P}^i = P^i \quad , \quad \overline{P}^+ = P^+ \quad , \quad \overline{P}^- = \frac{\alpha}{P^+}\,(L_0^{\rm T}+L_0^{\rm L}-1) \quad ,$$

where L_n^{T} and L_n^{T} are defined by (2) and (8), respectively. Since since for all α_m^μ operators $:\alpha_m^\mu \alpha_n^\nu :=: \overline{\alpha}_m^\mu\, \overline{\alpha}_n^\nu :$, the Lorentz generators can be written as

$$\overline{M}^{\mu\nu} = \overline{P}^\mu \overline{x}^\nu - \overline{P}^\nu \overline{x}^\mu + i \sum_{n=1}^{+\infty} \frac{1}{n} (\overline{\alpha}_{-n}^\mu \overline{\alpha}_n^\nu - \overline{\alpha}_{-n}^\nu \overline{\alpha}_n^\mu) \quad . \tag{38}$$

Calculating $\overline{\alpha}_n^\mu$ and $\overline{x}^\mu$ one obtains

$$\begin{aligned} \overline{x}^i &= x^i \quad , \quad \overline{x}^+ = 0 \quad , \quad \overline{x}^- = x^- \quad , \\ \overline{\alpha}_n^i &= \alpha_n^i \quad , \quad \overline{\alpha}_n^+ = 0 \quad , \quad \overline{\alpha}_n^- = \frac{\sqrt{\alpha}}{P^+} (L_n^{\mathrm{T}} + L_n^{\mathrm{L}} - 1) \quad . \end{aligned}$$

Substituting in (38) one gets the formulae (5), which completes the proof of equivalence. Our considerations can be summarized in the following statement.

The critical massive string model with the "Liouville momenta" q is isomorphic to the noncritical light-cone string model with the physical intercept $\alpha(0) = \frac{D-1}{24} - \frac{1}{2}q^2$.

The light-cone slice $\mathcal{H}_{\mathrm{lc}} \subset \mathcal{H}$ can be also defined as the subspace of all physical states Ψ satisfying the quantum light-cone gauge conditions $k \cdot \alpha_n \Psi = \alpha_n^+ \Psi = 0\,,\ n > 0$, known from the critical Nambu-Goto string. Let us stress that in the case of massive string one cannot impose these gauge conditions in the classical theory, because the Poisson bracket algebra of classical constraints develops a central extension [17]. On the quantum level the critical massive model is the only one where these conditions can be consistently imposed to remove redundancy related to the null states.

6 Nambu-Goto Gauge

Let us consider the subspace $\mathcal{H}_{\mathrm{NG}} \subset \mathcal{H}$ consisting of all excited physical states generated by the transverse A^i (A.1) and the longitudinal Brower $\widetilde{B}$ (A.2) DDF operators. The states from $\mathcal{H}_{\mathrm{NG}}$ do not contain excitation in the Liouville sector of the model. One can diagonalize the subalgebra generated by A^i and $\widetilde{B}$ by introducing [3]

$$B_n^{\mathrm{NG}} = \widetilde{B}_n - \mathcal{L}_n(A^i) + \delta_{n,0} \quad , \tag{39}$$

with

$$\mathcal{L}_n(A^i) = \tfrac{1}{2} \sum_{k=-\infty}^{+\infty} : A_{-k}^i A_{n+k}^i : \quad .$$

The space $\mathcal{H}_{\mathrm{NG}}$ is isomorphic as a tensor product of Fock space (generated by A^i) and Verma module (generated by B^{NG}) with the Hilbert space of the noncritical Nambu-Goto string [3] with the intercept of leading Regge trajectory $\alpha(0) = \frac{D-1}{24} - \frac{1}{2}q^2$.

Using the momenta and level decomposition of the space of excited physical states and the algebra (A.5) one can prove by the standard counting argument that $\mathcal{H}_{\mathrm{NG}}$ is a good gauge slice. Another more geometrical proof consists in constructing $\mathcal{H}_{\mathrm{NG}}$ by a transformation of the light-cone slice along null direction. The space $\mathcal{H}_{\mathrm{LC}}$ has the structure of the Fock space generated by the transverse A^i and the Liouville C excitations. There is an equivalent description of this space as a tensor product of the Fock space generated by the transverse

excitations and the Verma module generated by the Virasoro algebra constructed from the Liouville modes [19]:

$$\mathcal{L}_n(C) = +\frac{1}{2}\sum_{m=-\infty}^{+\infty} :C_{-m}C_{n+m}: + 2i\sqrt{\beta}nC_n + 2\beta\delta_{n,0} \quad .$$

Any state from $\mathcal{H}^{(N)}_{\rm LC}(p)$ can be written as an element of the Verma module created out of the highest weight vacuum vector :

$$\mathcal{H}^{(N)}_{\rm LC}(p) \ni \Psi = \vartheta^N(\mathcal{L}(C), A)\Omega_{p+\frac{\sqrt{\alpha}N}{k\cdot p}k} \quad .$$

Let us notice that the Brower $B^{\rm NG}_n$ (39) DDF operators differ from the Virasoro generators $\mathcal{L}_n(C)$ by the shifted Brower modes B_n (A.6)

$$B^{\rm NG}_n - \mathcal{L}_n(C) = B_n \quad , \quad n \in \mathbb{Z} \quad .$$

Since $[B_n, \mathcal{L}_n(C)] = 0$ and the central charge of the shifted Brower modes is zero (31), the map defined level by level by

$$\mathcal{H}^{(N)}_{\rm LC}(p) \ni \vartheta^N(\mathcal{L}(C), A)\Omega_{p+\frac{\sqrt{\alpha}N}{k\cdot p}k} \mapsto \vartheta^N(B^{\rm NG}, A)\Omega_{p+\frac{\sqrt{\alpha}N}{k\cdot p}k} \in \mathcal{H}^{(N)}_{\rm NG}(p) \quad , \tag{40}$$

is an isomorphism of Hilbert spaces preserving the structure of tensor product of Fock space and Verma module. For each state in $\mathcal{H}_{\rm LC}$ the map (40) is a shift in null direction. The image $\mathcal{H}_{\rm NG}$ is therefore a good gauge slice.

In contrast to the light-cone gauge $\mathcal{H}_{\rm LC}$ the subspace $\mathcal{H}_{\rm NG}$ is stable with respect to the Poincare transformations in $\mathcal{H}$. The induced representation of the Poincare algebra is simply given by the restriction of the generators M (28) on $\mathcal{H}$ to the subspace $\mathcal{H}_{\rm NG}$ and is identical with the representation one obtains in the covariant quantization of noncritical Nambu-Goto string with the physical intercept $\alpha(0) = \frac{D-1}{24} - \frac{1}{2}q^2$. One gets therefore the following result:

The critical massive string model with the "Liouville momentum" q is equivalent to the non-critical Nambu-Goto string model with the physical intercept $\alpha(0) = \frac{D-1}{24} - \frac{1}{2}q^2$.

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Appendix

In order to define the DDF operators one introduces the fields:

$$X^\mu(\theta) \;=\; q^\mu_0 + \alpha^\mu_0\theta + \sum_{m\neq 0}\frac{i}{m}\alpha^\mu_m {\rm e}^{-im\theta} \quad ,$$

$$\begin{aligned}
\Phi(\theta) &= \sum_{m\neq 0} \frac{i}{m}\beta_m \mathrm{e}^{-im\theta} \quad , \\
P^\mu(\theta) &= X^{\mu'}(\theta) = \sum_{m=-\infty}^{+\infty} \alpha_m^\mu \mathrm{e}^{-im\theta} \quad , \\
\Pi(\theta) &= \Phi'(\theta) = \sum_{m=-\infty}^{+\infty} \beta_m \mathrm{e}^{-im\theta} \quad ,
\end{aligned}$$

where $q_0^\mu = \sqrt{\alpha}x^\mu, \alpha_0^\mu = \frac{1}{\sqrt{\alpha}}P^\mu$, with the following commutation rules:

$$\begin{aligned}
[X^\mu(\theta), X^\nu(\theta')] &= -2\pi i\eta^{\mu\nu}\epsilon(\theta-\theta') \quad , \\
[\Phi(\theta), \Phi(\theta')] &= -2\pi i(\epsilon(\theta-\theta') + \epsilon(\theta'-\theta)) \quad , \\
[P_\mu(\theta), X^\nu(\theta')] &= -2\pi i\delta_\mu^\nu\delta(\theta-\theta') \quad , \\
[\Pi(\theta), \Phi(\theta')] &= -2\pi i(\delta(\theta-\theta') - 1) \quad , \\
[P_\mu(\theta), P_\nu(\theta')] &= -2\pi i\eta_{\mu\nu}\delta'(\theta-\theta') \quad , \\
[\Pi(\theta), \Pi(\theta')] &= -2\pi i\delta'(\theta-\theta') \quad .
\end{aligned}$$

The definitions of the transverse

$$A_m^i(k) = \frac{1}{2\pi}\int_0^{2\pi} d\theta : e_i \cdot P(\theta)\mathrm{e}^{\frac{imkX(\theta)}{k\cdot\alpha_0}} : \quad , \quad i \;=\; 1, ..., D-2 \quad . \tag{A.1}$$

and of the longitudinal

$$\widetilde{B}_m = \frac{1}{2\pi}\int_0^{2\pi} d\theta : \left(k\cdot\alpha_0\; k'\cdot P(\theta) - \frac{im}{2}\log'(\frac{k\cdot P(\theta)}{k\cdot\alpha_0})\right)\mathrm{e}^{\frac{imkX(\theta)}{k\cdot\alpha_0}} : \quad . \tag{A.2}$$

DDF operators are only slight modifications of the standard constructions of [18], and [3]. The "Liouville" DDF operators corresponding to the β excitations are defined by [16]:

$$C_m = \frac{1}{2\pi}\int_0^{2\pi} d\theta : \left(\Pi(\theta) - 2\sqrt{\beta}\log'(\frac{k\cdot P(\theta)}{k\cdot\alpha_0})\right)\mathrm{e}^{\frac{imkX(\theta)}{k\cdot\alpha_0}} : \quad . \tag{A.3}$$

The additional family of F operators generating non-physical states is defined by [22]

$$F_m(k) = \frac{1}{2\pi}\int_0^{2\pi} d\theta \; : \mathrm{e}^{imk\cdot X(\theta)} : \quad . \tag{A.4}$$

In contrast to the DDF operators $F_m(k)$ do not commute with the constraints

$$\begin{aligned}
[L_m, F_n(k)] &= -mF_n^m(k) \quad , \\
F_m^n(k) &= \frac{1}{2\pi}\int_0^{2\pi} d\theta\, \mathrm{e}^{in\theta} : \mathrm{e}^{imk\cdot X(\theta)} : \quad .
\end{aligned}$$

The algebra of the DDF operators reads:

$$\begin{aligned}
[A^i_m(k), A^j_n(k)] &= m\delta^{ij}\delta_{m,-n} \quad , \\
[C_m(k), C_n(k)] &= m\delta_{m,-n} \quad , \\
[\widetilde{B}_m(k), \widetilde{B}_n(k)] &= (n-m)\widetilde{B}_{n+m}(k) + 2n^3\delta_{m,-n} \quad , \\
[\widetilde{B}_n(k), A^i_m(k)] &= -mA^i_{m+n}(k) \quad , \\
[\widetilde{B}_n(k), C_m(k)] &= -mC_{n+m}(k) + 2in^2\sqrt{\beta}\delta_{n,-m} \quad .
\end{aligned} \tag{A.5}$$

The algebra above can be diagonalised by introducing the shifted Brower operators [3], [17]:

$$\widetilde{B}_n \longrightarrow B_n = \widetilde{B}_n - \mathcal{L}_n + \delta_{n,0} \tag{A.6}$$

where

$$\mathcal{L}_n = \frac{1}{2}\sum_{m=-\infty}^{+\infty} : A^i_{-m} \cdot A^i_{n+m} : + \frac{1}{2}\sum_{m=-\infty}^{+\infty} : C_{-m}C_{n+m} : + 2i\sqrt{\beta}C_n + 2\beta\delta_{n,0}. \tag{A.7}$$

For the new basis $\{A^i, B, C\}$ of the DDF operators one gets the "diagonal" commutation relations (31).

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